Calculus Review

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Our goal for this document will be to review some essential material from Calculus relevant to this class. These topics are usually all contained in a standard Calculus I or AP Calculus AB class, with a few exceptions. Limits will not be treated here, but we will start from derivatives and finish with integrals.

1. Derivatives

1.1. Definition.

Let $f(x)$ be a function on an interval $a \leq x \leq b$.

Definition 1.1 (Derivative)**.** Let c be a number with $a < c < b$. The *derivative of f at c* is defined to be the limit

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

if it exists. If this limit exists, we call it $f'(c)$.
If f has a derivative at every number between a and b, then we say f is *differentiable* on the interval $a < x < b$. In this case, $f'(x)$ is a function on the interval $a < x < b$. An alternative notation for $f'(x)$ is $\dfrac{df}{dx}$.

Remark 1.2. The fraction

$$\frac{f(x+h) - f(x)}{h}$$

is known as the *difference quotient* of h.

Let's begin by doing some basic examples

Example 1.3.

(1) $f(x) = c$, *c a constant*

(2) $g(x) = x$

(3) $j(x) = x^2$

(4) $k(x) = x^n$

Solution.

(1) The derivative is

$$f'(x) = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

(2) The derivative is

$$g'(x) = \lim_{h \to 0} \frac{(x+h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

(3) The derivative is

$$j'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{(x^2+2hx+h^2) - x^2}{h} = \lim_{h\to 0} \frac{2hx+h^2}{h} = \lim_{h\to 0}(2x+h) = 2x$$

(4) To do this one, let's first take a look at $(x+h)^n$. By the Binomial Theorem, we have

$$(x+h)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1}\binom{n}{n}h^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}h^k$$

For us, it's only important to know that $\binom{n}{0} = 1$ and $\binom{n}{1} = n$, but in general,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Thus,

$$(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \cdots$$

So, we then have that

$$\begin{aligned} k'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \cdots - x^n}{h} \\ &= \lim_{h\to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \cdots}{h} \\ &= \lim_{h\to 0}\left[nx^{n-1} + \binom{n}{2}x^{n-2}h^1 + \binom{n}{3}x^{n-3}h^2 + \cdots\right] = nx^{n-1} \end{aligned}$$

Let's close this section with a less trivial example. First recall the important limits

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

Example 1.4. *Find the derivative of $f(x) = \sin x$.*

Solution. Recall the sum formula for sine:

$$\sin(a+b) = \sin a \cos b + \cos a \sin b.$$

Writing down the difference quotient for $\sin(x)$ gives:

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\sin(x+h) - \sin x}{h} = \frac{\sin x\cos h + \cos x \sin h - \sin x}{h} \\ &= \frac{\sin x\,(\cos h - 1) + \cos x \sin h}{h} = \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\frac{\sin h}{h} \end{aligned}$$

So now, taking the limit as $h \to 0$ gives

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \frac{\sin h}{h}\right] \\
&= \lim_{h\to 0} \sin x \left(\frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \cos x \frac{\sin h}{h} \\
&= \sin x \lim_{h\to 0} \left(\frac{\cos h - 1}{h}\right) + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= (\sin x)(0) + (\cos x)(1) = \cos x
\end{aligned}$$

1.1.1. *Exercises.* Use the limit definition to find the derivatives of the following functions:

(1) x^3

(2) $4x^2 - 3x + 1$

(3) $\frac{1}{x}$

(4) $\sqrt{x}$

(5) $\cos x$

1.2. **Properties of Derivatives.**

We have already established two properties of derivatives in Example 1.3:

Theorem 1.5 (Constant Rule)**.** *For any real number c*

$$\frac{d}{dx}[c] = 0.$$

Theorem 1.6 (Power Rule)**.** *Let $n \neq 0$ be any real number. Then,*

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

Remark 1.7. In Example 1.3, we showed it for the case when n is a positive integer, but to prove it for real numbers in general requires knowledge of some other derivatives.

There are some more basic properties we can obtain rather easily. The next two theorems, together, say that the derivative is an *linear operator*:

Theorem 1.8 (Constant Multiple Rule)**.** *Let $f(x)$ be a differentiable function and let c be a constant. Then*

$$\frac{d}{dx}[cf(x)] = cf'(x).$$

Proof.

$$\begin{aligned}
\frac{d}{dx}[cf(x)] &= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h\to 0} c\left(\frac{f(x+h) - f(x)}{h}\right) \\
&= c\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = cf(x)
\end{aligned}$$

□

Theorem 1.9 (Sum and Difference Rules)**.** *Let $f(x)$ and $g(x)$ be two differentiable functions. Then*

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$
$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

1.2.1. *Exercises.* Compute the derivatives of the following functions

(1) $x^2 + x - 3$

(2) $5 - \frac{2}{3}x$

(3) $x^3 + x^2$

(4) x^7

(5) $\frac{1}{x^5}$

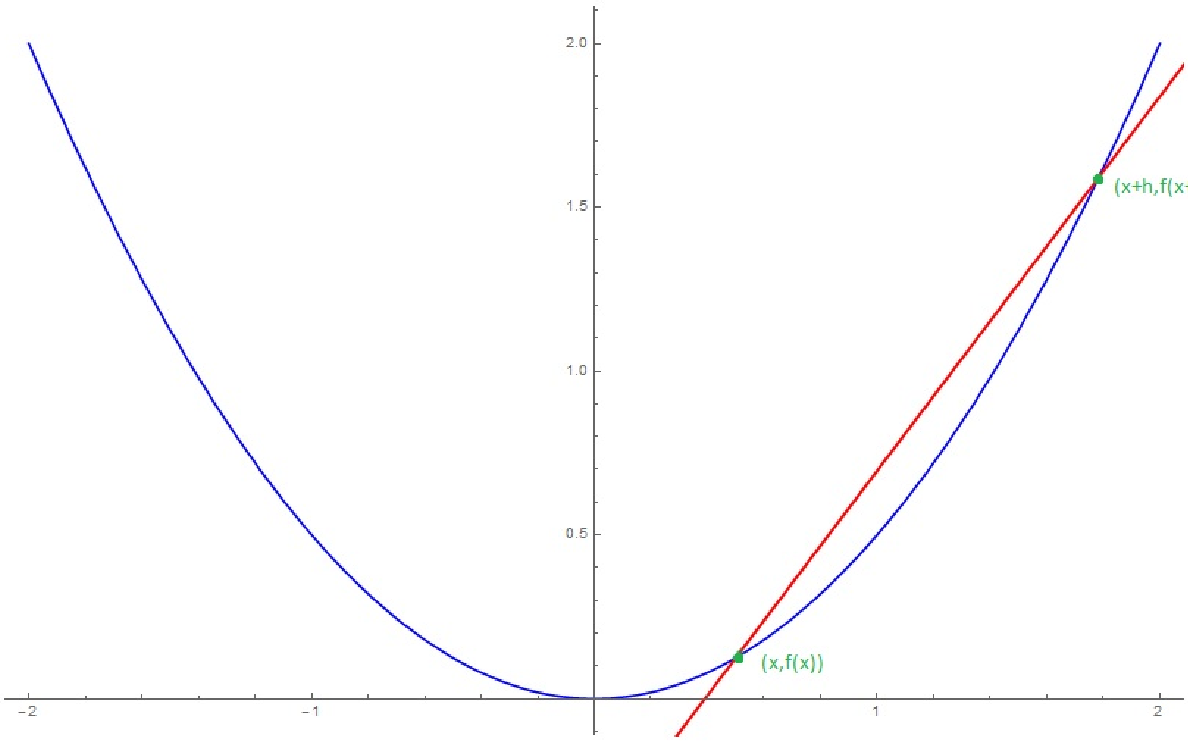
(6) $\sqrt[5]{t}$

(7) $x^2 - \frac{1}{3}\sin t$

(8) $\frac{5}{(2x)^3} + 2\cos t$

(9) $\frac{4x^3 + 2x + 5}{x}$

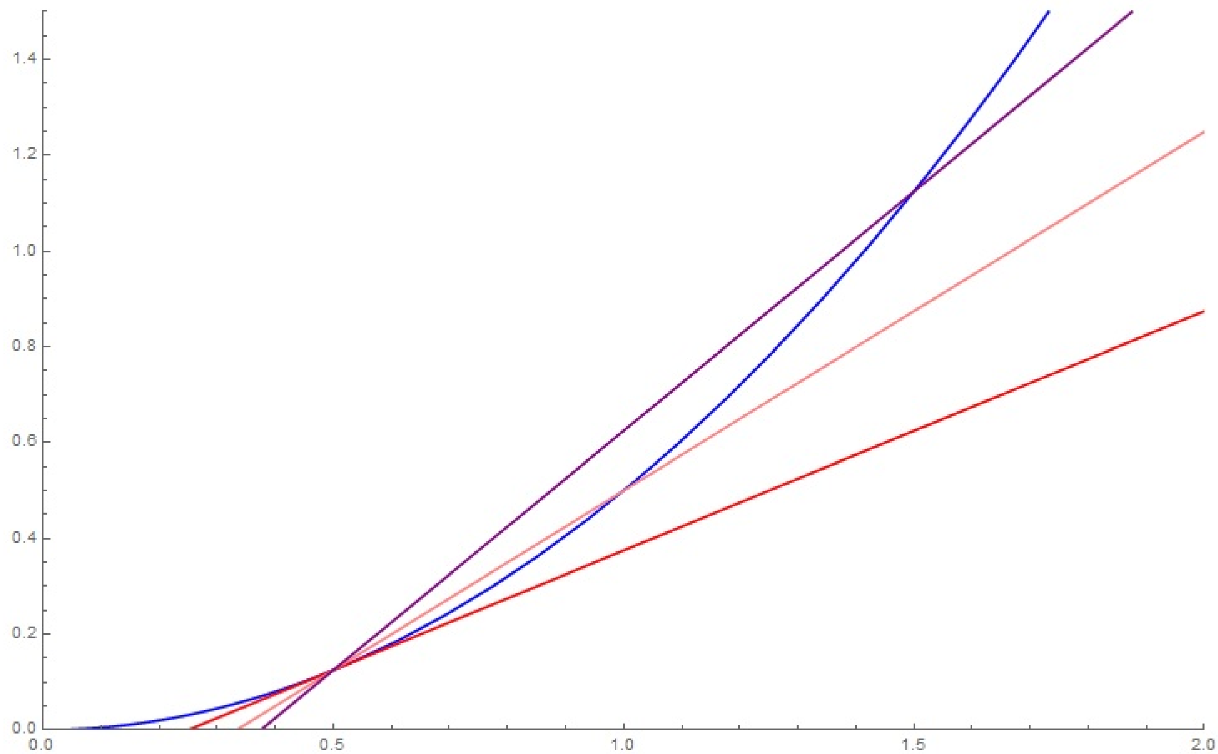
1.3. **Geometric Meaning of the Derivative.** Let $f(x)$ be a continuous function. Consider the line passing through the points $(x, f(x))$ and $(x+h, f(x+h))$. This line is called a *secant* of the graph.



The slope of the secant line is

$$\frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h},$$

the difference quotient. As h gets closer to 0, the point $(x+h, f(x+h))$ gets closer to the point $(x, f(x))$, and so the secant line approaches the tangent line. The red line is the tangent line and the pink and purple lines are secants. The pink line has a smaller h value than the purple line.



So, we then see that the slope of the tangent line is given by

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

i.e., the slope of the tangent line is given by the derivative.

Example 1.10. *Find an equation for the tangent line to the graph of* $f(x) = \frac{1}{8}x^4 + x$ *at the point* $(2, 4)$*.*

Solution. First, we take the derivative of $f(x)$

$$f'(x) = \frac{1}{8}(4x^3) + 1 = \frac{1}{2}x^3 + 1$$

Then, the slope of the tangent line at $(2, 4)$ is

$$f'(2) = \frac{1}{4}2^3 + 1 = 3$$

Using the point-slope form of a line, we find an equation for the tangent line to be

$$y - 4 = 3(x - 2)$$

Notice that we can use the derivative to check whether the function is increasing or decreasing by looking at it's tangent line. If the tangent line has positive slope, we can see that the function is increasing and if the tangent line has negative slope, then the function is decreasing. If the tangent line has 0 slope somewhere, i.e., the tangent line is horizontal, the function isn't changing at that point. That is, we have the correspondence

Value of $f'(x)$	Property of $f(x)$
Positive	Increasing
Negative	Decreasing
Zero	Value is not Changing

1.3.1. *Exercises.* Find an equation for the tangent line to the graph of the given function at the indicated point.

(1) $f(x) = x^2 - 9$, $(2, -5)$

(2) $g(x) = x^3$, $(0, 0)$

(3) $f(x) = x^3 + x^2 + 1$, $(2, 13)$

(4) $g(x) = \sqrt{x-1}$, $(5, 2)$

(5) $k(t) = 2\cos t$, $(\pi, -2)$

Find the points at which the following functions have horizontal tangent lines.

(6) $y = x^3 - x$

(7) $y = x + \sin x$

(8) $y = \sqrt{3}x + 2\cos t$

1.4. Product and Quotient Rule.

Given the machinery we have so far, differentiating something line $x \sin x$ will prove somewhat difficult. So, we need to establish some kind of rule for product of differentiable functions.

Theorem 1.11 (Product Rule)**.** *Let $f(x)$ and $g(x)$ be differentiable functions. Then*

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Unlike the proofs of the constant rule and sum and difference rules, the proof of the product rule is a bit less straightforward... We use a standard trick in analysis: adding and subtracting the same quantity (see the terms in green).

Proof.

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\left[\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h}+\frac{f(x+h)g(x)-f(x)g(x)}{h}\right]\\
&= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h}+\frac{f(x+h)-f(x)}{h}g(x)\right]\\
&= \lim_{h\to 0}f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x)\\
&= f(x)g'(x)+f'(x)g(x)
\end{aligned}$$

□

Let's do some examples now

Example 1.12. *Find the derivative of the following functions*

(1) $x\sin x$

(2) $(3x-2x^2)(5+4x)$

Solution.

(1) $\frac{d}{dx}[x\sin x]=\frac{d}{dx}[x]\sin x+x\frac{d}{dx}[\sin x]=\sin x+x\cos x$

(2)

$$\begin{aligned}
\frac{d}{dx}[(3x-2x^2)(5+4x)] &= \frac{d}{dx}[3x-2x^2](5+4x)+(3x-2x^2)\frac{d}{dx}[5+4x]\\
&= (3-4x)(5+4x)+(3x-2x^2)(4)\\
&= 15-20x+12x-16x^2+12x-8x^2=15+4x-24x^2
\end{aligned}$$

Now, we move on to the Quotient Rule

Theorem 1.13 (Quotient Rule)**.** *Let $f(x)$ and $g(x)$ be differentiable functions and assume that $g(x)\neq 0$. Then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

The proof is similar to that of the product rule. Instead, let's just move on to examples:

Example 1.14. *Find the derivative of the following functions*

(1) $\frac{x}{x^2+1}$

(2) $\frac{\sin t}{\sqrt{t}}$

Solution.

(1)

$$\begin{aligned}\frac{d}{dx}\left[\frac{x}{x^2+1}\right] &= \frac{\frac{d}{dx}[x](x^2+1) - x\frac{d}{dx}[x^2+1]}{(x^2+1)^2} \\ &= \frac{(1)(x^2+1)-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}\end{aligned}$$

(2)

$$\begin{aligned}\frac{d}{dt}\left[\frac{\sin t}{\sqrt{t}}\right] &= \frac{\frac{d}{dt}[\sin t]\sqrt{t} - \sin t\frac{d}{dt}[\sqrt{t}]}{(\sqrt{t})^2} \\ &= \frac{\cos t(\sqrt{t}) - (\sin t)\frac{1}{2\sqrt{t}}}{t} = \frac{2t\cos t - \sin t}{2t^{3/2}}\end{aligned}$$

Now that we have the quotient rule, we can find the derivatives of the other 4 trig functions.

$$\begin{aligned}\frac{d}{dx}[\tan x] &= \sec^2 x & \frac{d}{dx}[\cot x] &= -\csc^2 x \\ \frac{d}{dx}[\sec x] &= \sec x\tan x & \frac{d}{dx}[\tan x] &= -\csc x\cot x\end{aligned}$$

We will find the derivative of tangent, and the rest will be left as exercises

$$\begin{aligned}\frac{d}{dx}[\tan x] &= \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] \\ &= \frac{\frac{d}{dx}[\sin x]\cos x - \sin x\frac{d}{dx}[\cos x]}{(\cos x)^2} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \\ &= \sec^2 x\end{aligned}$$

1.4.1. *Exercises.* Compute the derivative of the following functions

(1) $(x^2+3)(x^2-4x)$

(2) $\sqrt{s}(s^2+8)$

(3) $(x^3-x)(x^2+2)(x^2+x-1)$

(4) $x^7\sin x$

(5) $\cos x\tan x$

(6) $\dfrac{3x^2-1}{x-9}$

(7) $\dfrac{x^2+5x+6}{x^2-16}$

(8) $\dfrac{3-\frac{1}{p}}{p-4}$

(9) $\csc t$

(10) $\cot t$

(11) $\sec t$

(12) $x^2\left(\dfrac{2}{x}-\dfrac{1}{x+1}\right)$

(13) $\dfrac{3(1-\cos\theta)}{\sin\theta}$

1.5. **The Chain Rule.** In this section, we cover one of the most important theorems about derivatives. The product rule and the chain rule together make up the two most important theorems of derivatives.

Theorem 1.15 (Chain Rule)**.** *Let $f(x)$ and $g(x)$ be differentiable functions such that the range of $g(x)$ lies in the domain of $f(x)$. Then*

$$\frac{d}{dx}[(f\circ g)(x)] = \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

The proof of this theorem uses another standard trick of analysis, multiplying by something over itself (see the green terms).

Remark 1.16. An alternative way to write down the derivative, $f'(c)$, is

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

Proof. In this proof, we assume that $g(x)\neq g(c)$ for $x\neq c$. It is, of course, possible to remedy this problem using the differentiability of f and g.

$$\begin{aligned}
\frac{d}{dx}[f(g(c))] &= \lim_{x\to c}\frac{f(g(x))-f(g(c))}{x-c}\\
&= \lim_{x\to c}\frac{f(g(x))-f(g(c))}{x-c}\,\frac{g(x)-g(c)}{g(x)-g(c)}\\
&= \lim_{x\to c}\frac{f(g(x))-f(g(c))}{g(x)-g(c)}\,\frac{g(x)-g(c)}{x-c}\\
&= \lim_{x\to c}\frac{f(g(x))-f(g(c))}{g(x)-g(c)}\,\lim_{x\to c}\frac{g(x)-g(c)}{x-c}\\
&= f'(g(c))g'(c)
\end{aligned}$$

□

Example 1.17. *Find the derivative of the function*

$$y=\cos(x^2).$$

Solution. We write $y = f(g(x))$ and see that $f(x) = \cos x$ and $g(x) = x^2$. Then, the chain rule gives

$$\frac{dy}{dx} = -\sin(x^2)(2x) = -2x\sin(x^2)$$

Using the chain rule, we can get the most general form of the power rule for derivatives

Theorem 1.18 (General Power Rule)**.** *Let $f(x)$ be a differentiable function and let $y = [f(x)]^n$, $n \neq 0$. Then*

$$\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$$

Example 1.19. *Compute the derivatives of the following functions*

(1) $(3x - 2x^2)^3$

(2) $\sin^3(4t^2)$

Solution.

(1) The inside function is $3x - 2x^2$, so

$$\frac{d}{dx}[(3x-2x^2)^3] = 3(3x-2x^2)^2\frac{d}{dx}[3x-2x^2] = 3(3x-2x^2)^2(3-4x) = (9-12x)(3x-2x^2)^2$$

(2) This function is a composition of three functions: the outside function is t^3, the middle function is $\sin t$, and the inside function is $4t^2$. Thus, we have

$$\begin{aligned}\frac{d}{dt}[\sin^3(4t^2)] &= 3\sin^2(4t^2)\frac{d}{dt}[\sin(4t^2)] \\ &= 3\sin^2(4t^2)\cos(4t^2)\frac{d}{dt}[4t^2] \\ &= 3\sin^2(4t^2)\cos(4t^2)(8t) = 24t\sin^2(4t^2)\cos(4t^2)\end{aligned}$$

1.5.1. *Exercises.* Compute the derivative of the following functions

(1) $\sin 4x$

(2) $4\sec^2 x$

(3) $(4x-1)^3$

(4) $\sqrt{x^2 - 4x + 2}$

(5) $\sqrt[5]{9y^2 + 1}$

(6) $\tan^2 5\theta$

(7) $\frac{1}{2}x^2\sqrt{16 - x^2}$

(8) $\left(\frac{3x-1}{x^2+3}\right)^2$

(9) $(2+(x^2+1)^3)^2$

(10) $\cos(x\sin(x\tan x))$

1.6. **Summary of Derivative Rules.**

Let f and g be differentiable functions, and let c be a constant.

1.6.1. *Basic Differentiation Rules.*

Constant Multiple Rule:

$$\frac{d}{dx}[cf] = cf'$$

Sum/Difference Rule:

$$\frac{d}{dx}[f \pm g] = f' \pm g'$$

Product Rule:

$$\frac{d}{dx}[fg] = f'g + fg'$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'g - fg'}{g^2}$$

1.6.2. *Derivatives of Basic Functions.*

$$\begin{aligned} \frac{d}{dx}[c] &= 0 & \frac{d}{dx}[x^n] &= nx^{n-1}\ (n \neq 0) \\ \frac{d}{dx}[\sin x] &= \cos x & \frac{d}{dx}[\cos x] &= -\sin x \\ \frac{d}{dx}[\tan x] &= \sec^2 x & \frac{d}{dx}[\sec x] &= \sec x \tan x \\ \frac{d}{dx}[\cot x] &= -\cot^2 x & \frac{d}{dx}[\csc x] &= -\csc x \cot x \end{aligned}$$

1.6.3. *Chain Rule.*

Chain Rule:

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

General Power Rule:

$$\frac{d}{dx}[(f(x))^n] = n(f(x))^{n-1}f'(x),\ n \neq 0$$

1.7. Higher Derivatives.

Given a function $f(x)$, we can take its derivative $f'(x)$. If the limit

$$\lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}$$

exists, then we write $f''(x)$ and call it the *second derivative* of $f(x)$. If this second derivative exists, then we say that f is *twice differentiable*. We can, of course, continue in this way to take higher derivatives of f: third, fourth, fifth, etc... The notation for these are

$\#^{\text{th}}$ derivative	0	1	2	3	4	$\cdots$	n
notation	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$\cdots$	$f^{(n)}(x)$

1.8. Physical Interpretation of the Derivative.

Let's suppose that we have a function $x = x(t)$ which describes the position at time t of a particle moving around the x-axis. Recall that *velocity* is the instantaneous rate of change in position with respect to time. If we let the initial time be t and $t + h$ some time shortly in the future, then $\Delta t = (t+h)-t = h$ is the change in time and $\Delta x = x(t+h)-x(t)$ is the change in position during that time. So, the average velocity over that period of time is given by

$$v_{avg} = \frac{\Delta x}{\Delta t}.$$

If we let this change in time go to zero (i.e., $\Delta t \to 0$, which is equivalent to $h \to 0$), we get that the instantaneous velocity at time t is

$$v(t) = \lim_{\Delta t\to 0} \frac{\Delta x}{\Delta t} = \lim_{h\to 0} \frac{x(t+h)-x(t)}{h} = x'(t)$$

It is actually customary when dealing with functions of time to denote derivatives with a dot over the function, i.e.,

$$v(t) = \dot{x}(t)$$

If we combine this with our knowledge of the geometric meaning of the derivative, we can see the following: Plot the function $x = x(t)$ and look at the slope of the tangent line. If the slope of the line is very steep this means the derivative is very large (positive or negative large number). Interpreting what that means in terms of position and time, if the tangent line is steep, that means that the position has changed a large amount in a small amount of time, i.e., that the velocity is large. Likewise, we can associate a smaller slope with a smaller velocity since the change in position with respect to time will be smaller.

Finally, recall that *acceleration* is the change in velocity with respect to time. Following a similar argument to the one for velocity, we find that acceleration is the derivative of velocity, i.e.,

$$\begin{aligned} a(t) &= \dot{v}(t) = \ddot{x}(t) \\ &= v'(t) = x''(t) \end{aligned}$$

1.9. Important Theorems about Derivatives.

Theorem 1.20 (Rolle's Theorem). *Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a) = f(b)$, then there is at least one number c in (a,b) such that $f'(c) = 0$.*

Theorem 1.21 (Mean Value Theorem)**.** *If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. The secant line between the points $(a, f(a))$ and $(b, f(b))$ on the graph is given by

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

Let $g(x)$ be the difference between $f(x)$ and y

$$\begin{aligned} g(x) &= f(x) - y \\ &= f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a) \end{aligned}$$

The continuity of f on $[a, b]$ and differentiability of f on (a, b) imply the same properties for $g(x)$. Notice that $g(a) = 0$ and $g(b) = 0$, so that we may apply Rolle's Theorem to find a c in the interval (a, b) such that $g'(c) = 0$. But

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

so that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

which implies

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

□

2. Integration

2.1. Antiderivatives and Indefinite Integrals.

An antiderivative, as its name implies, is the opposite of a derivative. It is the inverse operation to the derivative.

Definition 2.1 (Antiderivative)**.** Let $f(x)$ and $F(x)$ be functions. We say that F is an *antiderivative* of f if

$$\frac{d}{dx}[F(x)] = f(x)$$

The antiderivative of a function is not unique!!

Example 2.2. *Find two antiderivatives of $f(x) = x^2$.*

Solution. We know that

$$\frac{d}{dx}[x^3] = 3x^2$$

so

$$\frac{d}{dx}\left[\frac{1}{3}x^3\right] = x^2.$$

Thus $F(x) = \frac{1}{3}x^3$ is an antiderivative of x^2. Notice that $G(x) = \frac{1}{3}x^3 + 2$ is also an antiderivative of f since

$$\frac{d}{dx}\left[\frac{1}{3}x^3 + 2\right] = x^2 + 0 = x^2.$$

The most general antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3 + C$. We call this the *indefinite integral* of $f(x) = x^2$.

Definition 2.3 (Indefinite Integral)**.** Let $f(x)$ be a function and let $F(x)$ be any antiderivative of $f(x)$. Then the *indefinite integral* of $f(x)$ is the function

$$\int f(x)\,dx = F(x) + C$$

where C is an arbitrary constant known as the *constant of integration*.

Remark 2.4. In $\int f(x)\,dx$, the function $f(x)$ is called the *integrand*.

Integration and differentiation are inverses to each other:

$$\begin{aligned} \int f'(x)\,dx &= f(x) + C \\ \frac{d}{dx}\left[\int f(x)\,dx\right] &= f(x) \end{aligned}$$

Example 2.5. *Compute the indefinite integral of the following functions*

(1) $x + 7$

(2) $\sqrt{t}$

(3) $4\sin\theta$

Solution.

(1) Since the derivative of x^2 is $2x$, we get that an antiderivative of x is $\frac{1}{2}x^2$. Likewise, since the derivative of $7x$ is 7, an antiderivative of 7 is $7x$. Thus the indefinite integral is

$$\int (x+7)dx = \frac{1}{2}x^2 + 7x + C$$

(2) Recall that $\sqrt{t} = t^{1/2}$. Since the derivative of $t^{3/2}$ is $\frac{3}{2}t^{1/2}$, an antiderivative of $\sqrt{t}$ is $\frac{2}{3}t^{3/2}$. Thus the indefinite integral is

$$\int \sqrt{t}\, dt = \frac{2}{3}\sqrt{t} + C$$

(3) Recall that the derivative of $\cos\theta$ is $-\sin\theta$. Thus, an antiderivative of $4\sin\theta$ is $-4\cos\theta$. So, the indefinite integral is

$$\int 4\sin\theta\, d\theta = -4\cos\theta + C$$

2.1.1. *Exercises.* Compute the indefinite integral of the following functions. Check your answers by differentiating your result and matching it with the original function.

(1) $9t^2$

(2) $\sqrt[3]{x}$

(3) $\dfrac{x+6}{\sqrt{x}}$

(4) $\sec y(\tan y - \sec y)$

2.2. **Properties of Indefinite Integrals.** Since the derivative and indefinite integral are inverse operations to each other, we can reverse properties we've found for derivatives to get properties for integrals. Let C and k be constants.

Differentiation Formula	Integration Formula
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf'(x)\,dx = kf(x) + C$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int (f'(x) \pm g'(x))\,dx = f(x) \pm g(x) + C$
$\frac{d}{dx}[x^n] = nx^{n-1},\ n \neq 0$	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C,\ n \neq -1$ (Power Rule)
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$

Recall that $n \neq 0$ in the power rule for derivatives because $x^0 = 1$ (there's an issue if $x = 0$, but we'll just sweep that under the carpet and ignore it for sake of brevity), and so it is just the derivative of a constant, which is 0. This corresponds to the case when $n = -1$ for integrals, since an antiderivative of $\frac{1}{x}$ cannot possibly be just a constant function. We'll deal with the integral of $\frac{1}{x}$ later. Likewise, notice we don't have formulas for the integrals of $\tan x$, $\sec x$, $\cot x$, or $\csc x$. Computing these integrals requires some tricks and knowing what the integral of $\frac{1}{x}$ is.

2.2.1. *Exercises.* Compute the following indefinite integrals

(1) $\int (13 - x)dx$

(2) $\int (x^5 + 1)dx$

(3) $\int (t^2 - \cos t)dt$

(4) $\int \frac{3}{x^7}dx$

(5) $\int (\theta^2 - \sec^2 \theta)d\theta$

(6) $\int (1 - \csc t \cot t)dt$

(7) $\int (y^{-3/2} + \tan^2 y + 1)dy$

2.3. Riemann Sums and the Definite Integral.

One goal of a definite integral is to find the area under a curve $y = f(x)$ over an interval $[a, b]$. Other uses of a definite integral include integrating a velocity function from the start time to the end time to find the total distance traveled.

2.3.1. *The Riemann Sum.* Suppose that f is a function defined on the closed interval $[a, b]$, and let n be a positive integer. Partition the interval $[a, b]$ into n parts

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

and let $\Delta x_i = x_i - x_{i-1}$ for $1 \leq i \leq n$, i.e., Δx_i is the length of the subinterval $[x_{i-1}, x_i]$ for $i = 1, ..., n$.

Definition 2.6 (Riemann Sum)**.** Let x_i^* be any point in the interval $[x_{i-1}, x_i]$. The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

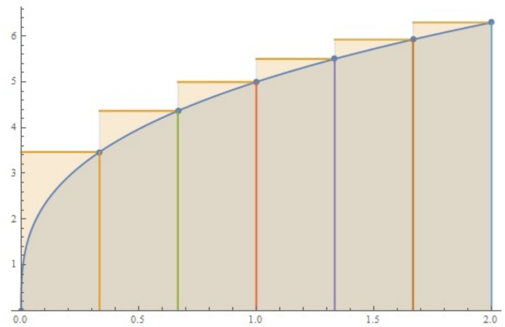
is called a *Riemann Sum*.

Let us observe what the Riemann sum is telling us in the case when $f(x) \geq 0$ for $a \leq x \leq b$: Over the interval $[x_{i-1}, x_i]$, we draw a rectangle of height $f(x_i^*)$, then the area of that rectangle is $f(x_i^*)\Delta x$. The Riemann sum is adding up the area of these rectangles, and thus giving an approximation of the area under the curve $y = f(x)$.

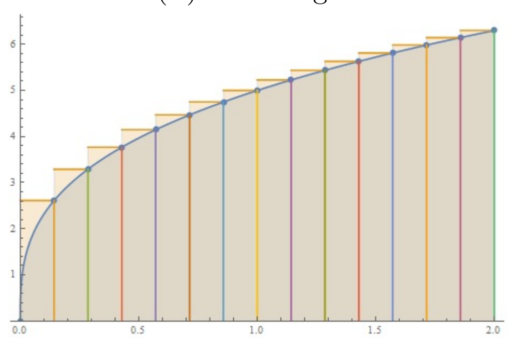
To this effect, it is common, and rather convenient, to choose the intervals $[x_{i-1}, x_i]$ to have equal length. To do this, just force

$$\Delta x = \frac{b - a}{n}$$

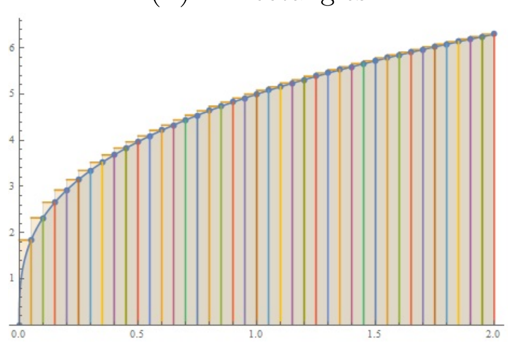
and let $x_i = a + i\Delta x$, $i = 0, 1, 2, ..., n$.



(A) 6 rectangles



(B) 14 rectangles



(C) 40 rectangles

FIGURE 1. Riemann sums using increasing numbers of rectangles become better approximations of the area under a positive function.

2.3.2. *Some Special Summations.*

(1) $\displaystyle\sum_{i=1}^{n} c = cn$, c is a constant

(2) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(3) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(4) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

2.3.3. *The Definite Integral.*

Definition 2.7 (Definite Integral)**.** Given a function f on an interval $[a, b]$ and a partition of $[a, b]$ as above with equal sized subintervals, we say that f is *integrable* on $[a, b]$ if the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

exists. In this case, we write

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

which we call the *definite integral* of f from a to b.

To facilitate being able to tell if a function is integrable, we have the following theorem

Theorem 2.8. *If f is a continuous function on the interval $[a, b]$, then f is integrable on $[a, b]$, i.e.,*

$$\int_a^b f(x)\,dx$$

exists.

Example 2.9. *Compute the following definite integrals:*

(1) $\displaystyle\int_0^3 x\,dx$

(2) $\displaystyle\int_{-1}^2 3x^2\,dx$

Solution.

(1) Using the conventions above, $\Delta x = \dfrac{b-a}{n} = \dfrac{3-0}{n} = \dfrac{3}{n}$. Then $x_i = a + i\Delta x = 0 + i\dfrac{3}{n} = \dfrac{3i}{n}$. For the sample points, x_i^*, we will choose the right endpoints of the subintervals $[x_{i-1}, x_i]$, i.e., $x_i^* = x_i = \dfrac{3i}{n}$. Then, we have

$$\begin{aligned}
\int_0^3 x\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{9i}{n^2} = \lim_{n\to\infty} \frac{9}{n^2} \sum_{i=1}^{n} i \\
&= \lim_{n\to\infty} \frac{9}{n^2}\frac{n(n+1)}{2} = \lim_{n\to\infty} \frac{9}{2}\frac{n^2+n}{n^2} \\
&= \lim_{n\to\infty} \frac{9}{2}\left(1 + \frac{1}{n}\right) = \frac{9}{2}
\end{aligned}$$

(2) This time, we have $\Delta x = \dfrac{2-(-1)}{n} = \dfrac{3}{n}$. This time $x_i = -1 + i\dfrac{3}{n}$. Once again, we will choose the right endpoints for x_i^* so that $x_i^* = x_i = -1 + \dfrac{3i}{n}$. So, the definite

integral is

$$\begin{aligned}
\int_{-1}^{2} 3x^2\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} 3\left(-1+\frac{3i}{n}\right)^2 \frac{3}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} 3\left(\frac{3}{n}-\frac{18i}{n^2}+\frac{27i^2}{n^3}\right) \\
&= \lim_{n\to\infty} 3\left(\frac{3}{n}\sum_{i=1}^{n}1-\frac{18}{n^2}\sum_{i=1}^{n}i+\frac{27}{n^3}\sum_{i=1}^{n}i^2\right) \\
&= \lim_{n\to\infty} 3\left(\frac{3}{n}n-\frac{18}{n^2}\frac{n(n+1)}{2}+\frac{27}{n^3}\frac{n(n+1)(2n+1)}{6}\right) \\
&= \lim_{n\to\infty} 3\left(3-9\frac{n^2+n}{n^2}+\frac{9}{2}\frac{2n^3+3n^2+n}{n^3}\right) \\
&= \lim_{n\to\infty} 3\left(3-9\left(1+\frac{1}{n}\right)+\frac{9}{2}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right) \\
&= 3(3-9+9)=9
\end{aligned}$$

2.3.4. *Exercises.* Compute the following definite integrals using Riemann sums

(1) $\int_0^3 x^3\,dx$

(2) $\int_1^4 (x^2+1)\,dx$

2.4. The Fundamental Theorem of Calculus.

As it stands, computing a definite integral is quite the chore! However, the following theorem turns it into a problem of finding an antiderivative, then plugging in some numbers!

Theorem 2.10 (Fundamental Theorem of Calculus)**.** *If a function f is continuous on the closed interval $[a,b]$ and F is an antiderivative of f on the interal $[a,b]$, then*

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

Proof. Let's take a partition of $[a,b]$

$$a=x_0<x_1<\cdots<x_n=b$$

and let $\Delta x_i = x_i - x_{i-1}$. We will cleverly rewrite $F(b)-F(a)$ as

$$F(b)-F(a)=F(x_n)-F(x_{n-1})+F(x_{n-1})-\cdots-F(x_1)+F(x_1)-F(x_0)=\sum_{i=1}^{n}\left(F(x_i)-F(x_{i-1})\right)$$

Using the mean value theorem, we know there is a c_i in each interval $[x_i,x_{i+1}]$ such that

$$F'(c_i)=\frac{F(x_i)-F(x_{i-1})}{x_i-x_{i-1}}$$

But $F'(c_i)=f(c_i)$ and since $\Delta x_i = x_i - x_{i-1}$ we have

$$f(c_i)\Delta x_i = F(x_i)-F(x_{i-1})$$

so that

$$F(b) - F(a) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

Now, taking $n \to \infty$, we arrive at

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

□

Remark 2.11. When evaluating a definite integral, a common notation to use is

$$\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a)$$

Example 2.12. *Compute the following definite integrals*

(1) $\displaystyle\int_{-1}^{2} 3x^2\,dx$

(2) $\displaystyle\int_{1}^{4} \frac{u-2}{\sqrt{u}}du$

Solution.

(1) An antiderivative of $3x^2$ is x^3, so

$$\int_{-1}^{2} 3x^2\,dx = x^3\big|_{-1}^{2} = 2^3 - (-1)^3 = 8 - (-1) = 9$$

(2) Since $\dfrac{u-2}{\sqrt{u}} = \sqrt{u} - 2u^{-1/2}$, we can find an antiderivative to be $\dfrac{2}{3}u^{3/2} - 4\sqrt{u}$. Then,

$$\begin{aligned}\int_{1}^{4} \frac{u-2}{\sqrt{u}}du &= \left(\frac{2}{3}u^{3/2} - 4\sqrt{u}\right)\bigg|_1^4 \\ &= \left(\frac{16}{3} - 8\right) - \left(\frac{2}{3} - 4\right) \\ &= \frac{14}{3} - 4 = \frac{2}{3}\end{aligned}$$

Now, once again, derivatives and integrals are inverse to each other, so we can rephrase the Fundamental Theorem of Calculus in an alternate way to reflect this.

Theorem 2.13 (Fundamental Theorem of Calculus (Alternate Version))**.** *If a function f is continuous on an open interval I containing a, then for every x in the interval I*

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Proof. Let F be an antiderivative of f, then the Fundamental Theorem of Calculus gives

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

Differentiating this gives

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}\left(F(x) - F(a)\right) = f(x)$$

□

2.4.1. *Exercises.* Compute the following definite integrals

(1) $\displaystyle\int_0^2 6x\,dx$

(2) $\displaystyle\int_{-1}^2 (7-2x)\,dx$

(3) $\displaystyle\int_0^\pi \cos t\,dt$

(4) $\displaystyle\int_0^\pi \sin t\,dt$

(5) $\displaystyle\int_0^2 \sqrt{x}(x+8)\,dx$

(6) $\displaystyle\int_1^8 \sqrt{\frac{2}{x}}\,dx$

(7) $\displaystyle\int_0^{\pi/4} \frac{1-\sin^2\theta}{\cos^2\theta}\,d\theta$

(8) $\displaystyle\int_{-1}^1 (\sqrt[3]{x}-2)\,dx$

(9) $\displaystyle\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,d\theta$

(10) $\displaystyle\int_{-\pi/2}^{\pi/2} (2x+\cos x)\,dx$

(11) $\displaystyle\int_{-1}^1 |x|\,dx$

(12) $\displaystyle\int_0^5 |x-2|\,dx$

(13) $\displaystyle\int_0^4 |x^2-4x+3|\,dx$

2.5. Integration by Substitution.

The technique of integration by substitution, also known more colloquially as *u-substitution* is the reversal of the Chain Rule.

Theorem 2.14 (Integration by Substitution)**.** *Let g be a differentiable function whose range is an interval I and let f be function continuous on the interval I. Let F be an antiderivative for f on I, then*

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C$$

Proof. Recall the Chain Rule

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x) = f(g(x))g'(x)$$

Integrating both sides of this equation gives

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C$$

□

Remark 2.15. In the spirit of the name u-substitution, let $u = g(x)$. Then $du = g'(x)\,dx$ and we can rewrite the theorem as

$$\int f(u)du = F(u) + C$$

The key to using this theorem to try and look for the $f(g(x))g'(x)$ pattern in the integrand. Let's do a few examples to see this

Example 2.16. *Compute the following integrals*

(1) $\int (x^2+1)^2(2x)\,dx$

(2) $\int 5\cos 5x\,dx$

(3) $\int x(x^2+4)^2\,dx$

Solution.

(1) Notice that if we take $u = x^2 + 1$, then $du = 2x\,dx$ and we can find both of these things in the integral

$$\int \underbrace{(x^2+1)^2}_{u^2}\overbrace{(2x)\,dx}^{du}$$

So, then we get

$$\int (x^2+1)^2(2x)\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(x^2+1) + C$$

It is VERY IMPORTANT that you do not forget to plug back in everywhere there is a u!!

(2) In this case, we can use $u = 5x$ and then $du = 5dx$. So,

$$\int 5\cos 5x\,dx = \int \cos u\,du = \sin u + C = \sin 5x + C$$

(3) What we want to choose for u is $u = x^2 + 4$, but $du = 2x\,dx$ and there is no 2 in the integrand... We can create one though by multiplying and dividing by 2 in the integrand

$$\int x(x^2+4)^2\,dx = \int \frac{1}{2}2x(x^2+4)^2\,dx = \frac{1}{2}\int 2x(x^2+4)^2\,dx$$

An alternative, and much easier way to accomplish this substitution is to just divide by 2 in the equation $du = 2x\,dx$ to get $\frac{1}{2}du = x\,dx$. Either way, we end up with

$$\int x(x^2+4)^2\,dx = \frac{1}{2}\int u^2\,du = \frac{1}{6}u^3 + C = \frac{1}{6}(x^2+4)^3 + C$$

As the last example showed, the integrands, as given, might not always line up well with the $f'(g(x))g'(x)$ form, and we have to mess with the integrand, or with the du to get the formula to work. This is a common issue, and sometimes it requires even more trickery! It could involve changing the original substitution into another form to solve the problem. This process is more appropriately called a *change of variables.*

Example 2.17. *Compute the following integrals*

(1) $\int x\sqrt{2x-4}\,dx$

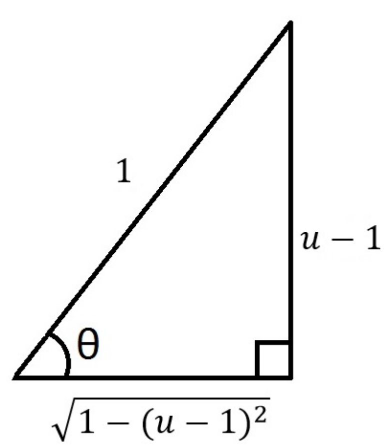
(2) $\int \sqrt{1+\sin\theta}\,d\theta$

Solution.

(1) Here, we let $u = 2x - 4$, then $du = 2\,dx$. Now, notice how that x outside the square root is not covered by the du... to take care of it, we can solve for x in our substitution $x = \frac{1}{2}u + 2$, then

$$\begin{aligned}\int x\sqrt{2x-4}\,dx &= \int \left(\frac{1}{2}u + 2\right)\sqrt{u}\frac{1}{2}du \\ &= \int \left(\frac{1}{4}u^{3/2} + \sqrt{u}\right) du = \frac{1}{10}u^{5/2} + \frac{2}{3}u^{3/2} + C \\ &= \frac{1}{10}(2x-4)^{5/2} + \frac{2}{3}(2x-4)^{3/2} + C\end{aligned}$$

(2) In this problem, the apparent substitution to make is $u = 1 + \sin\theta$. Then $du = \cos\theta\,d\theta$. However, there is nowhere in the integrand something we can use to take care of the $\cos\theta$. However, since we are dealing with trig functions, we can use $\sin\theta$ to find $\cos\theta$. Since $\sin\theta = u - 1$, we can draw the right triangle for the angle θ, then using the fact that $\sin\theta = \frac{\text{opposite edge}}{\text{hypotenuse}}$, we can find $\cos\theta = \frac{\text{adjacent edge}}{\text{hypotenuse}}$.



So, $\cos\theta = \sqrt{1-(u-1)^2} = \sqrt{2u-u^2}$. Thus, $d\theta = \dfrac{1}{\cos\theta}du = \dfrac{1}{\sqrt{2u-u^2}}du$ and so we can finally get

$$\int \sqrt{1+\sin\theta}\, d\theta = \int \sqrt{u}\frac{1}{\sqrt{2u-u^2}}du = \int \sqrt{\frac{1}{2-u}}du$$

Now, this integral requires a second substitution, $v = 2-u$, $dv = -du$. Doing this gives

$$\int \sqrt{\frac{1}{2-u}}du = \int v^{-1/2}(-dv) = -2\sqrt{v} + C = -2\sqrt{2-u} + C = -2\sqrt{1-\sin\theta} + C$$

To close out this section on integration by substitution, let's briefly discuss how to handle definite integrals. The only difference now is we have to change the bounds of the integral as well. The nice part is, if we change the bounds, we don't have to switch variables back at the end of the problem!

Theorem 2.18 (Change of Variables for Definite Integrals)**.** *Suppose the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g. Then,*

$$\int_a^b f(g(x))g'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

What this basically means is that we plug the old bounds into our u-substitution to get the new bounds.

Example 2.19. *Compute the definite integral*

$$\int_0^1 x(x^2+1)^3\, dx$$

Solution. We let $u = x^2+1$, then $du = 2x\, dx$ or $\dfrac{1}{2}du = x\, dx$. The bounds become $u(0) = 0^2+1 = 1$ for the lower bound and $u(1) = 1^2+1 = 2$ for the upper bound, and we get

$$\int_0^1 x(x^2+1)^3\, dx = \int_1^2 \frac{1}{2}u^3\, du = \frac{1}{8}u^4\Big|_1^2 = \frac{1}{8}(16-1) = \frac{15}{8}$$

2.5.1. *Exercises.* Compute the given integrals

(1) $\int (1+6x)^7(6)\, dx$

(2) $\int \sqrt[3]{3-4x^2}(-8x)\, dx$

(3) $\int x^2(6-x^3)^5\, dx$

(4) $\int t\sqrt{t^2+1}\, dt$

(5) $\int \sin\theta\cos\theta\, d\theta$

(6) $\int \sin^3\theta \cos\theta \, d\theta$

(7) $\int \left(1 + \frac{1}{s}\right)^3 ds$

(8) $\int \left(x^2 + \frac{1}{(3x)^2}\right) dx$

(9) $\int \frac{\sin x}{\cos^3 x} \, dx$

(10) $\int \frac{1}{q^2} \cos \frac{1}{q} \, dq$

(11) $\int x\sqrt{x+6} \, dx$

(12) $\int (x+1)\sqrt{2-x} \, dx$

(13) $\int \frac{x^2 - 1}{\sqrt{2x-1}} \, dx$

(14) $\int_{-1}^{1} x(x^2+1)^3 \, dx$

(15) $\int_0^1 x\sqrt{1-x^2} \, dx$

(16) $\int_0^4 \frac{1}{\sqrt{2x+1}} \, dx$

(17) $\int_1^5 \frac{x}{\sqrt{2x-1}} \, dx$

(18) $\int_1^9 \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2} \, dx$

2.6. **Integration by Parts.** (This will be skipped in class, but for sake of flow, convenience, and completion, it will be included here.)

We saw that reversing the Chain Rule gives Integration by Substitution. Now, what if we integrate the Product Rule? Recall that the product rule is

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

We integrate both sides

$$\int \frac{d}{dx}[f(x)g(x)] \, dx = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx$$

which leaves us with

$$f(x)g(x) = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx$$

where the constants of integration are implicit in the indefinite integrals. We often rewrite this and arrive at

Theorem 2.20 (Integration by Parts)**.** *Let f and g be functions whose derivatives are continuous. Then*

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Remark 2.21. If we let $u = f(x)$ and $v = g(x)$, then we can arrive at the commonly stated form of integration by parts

$$\int u\,dv = uv - \int v\,du$$

Note, in particular, that $dv = g'(x)dx$.

When using integration by parts, the question becomes about how to choose u and dv. The general idea is to choose u to be a function so that du is simpler than u, and to choose dv to be the most complicated part we can actually integrate. A rule for choosing u and dv which can help is to choose them according to the word LIATE (sometimes people write ILATE instead... it doesn't matter since if you only have functions corresponding to I and L, it's probably going to be a hard integral anyway). The acronym stands for

Logarithmic
Inverse Trigonometric
Algebraic
Trigonometric
Exponential

The general idea is to choose u to be the part of the integrand which shows up first in this list, and dv to be the rest. As of yet, we haven't discussed exponential, inverse trig, or logarithmic derivatives/integrals, but we can still get some early practice with the method.

Example 2.22. *Compute the following integral*

$$\int x\cos x\,dx$$

Solution. We choose $u = x$ and $dv = \cos x\,dx$ since then $du = dx$ and $v = \sin x$ (notice how du is simpler than u). Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

There is also a version of integration by parts for definite integrals. To obtain it, just integrate the product rule from a to b, and you will find the following

Theorem 2.23 (Integration by Parts for Definite Integrals)**.** *Let f and g be functions whose derivatives are continuous on (a, b). Then*

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

Example 2.24. *Compute*

$$\int_0^\pi x^2\sin x\,dx$$

Solution. Since we cannot integrate $x^2 \sin x$ using any of our other techniques, we will do integration by parts with $u = x^2$ and $dv = \sin x, dx$. Then $du = 2x\,dx$ and $v = -\cos x$, so

$$\int_0^\pi x^2 \sin x\,dx = -x^2 \cos x\Big|_0^\pi + 2\int_0^\pi x\cos x\,dx$$

Now, we again have to use integration by parts on $x\cos x$. Use $u = x$ and $dv = \cos x\,dx$, so that $du = dx$ and $v = \sin x$, then the integral becomes

$$\begin{aligned}
\int_0^\pi x^2 \sin x\,dx &= -x^2 \cos x\big|_0^\pi + 2\int_0^\pi x\cos x\,dx \\
&= -x^2 \cos x\big|_0^\pi + 2\left(x\sin x|_0^\pi - \int_0^\pi \sin x\,dx\right) \\
&= -x^2 \cos x\big|_0^\pi + 2\left(x\sin x|_0^\pi - (-\cos x)|_0^\pi\right) \\
&= (-\pi^2\cos\pi - 0) + 2\left[(\pi\sin\pi - 0) - (-\cos\pi + \cos 0)\right] \\
&= \pi^2 + 2\,[0 - 2] = \pi^2 - 4
\end{aligned}$$

3. Calculus of Transcendental Functions

3.1. Calculus of Inverse Functions.

Suppose that f is an invertible function with inverse f^{-1} and further suppose that both f and f^{-1} are differentiable. Then, we get the following interpretation of the derivative of f^{-1}

Theorem 3.1. *If f is a differentiable function and f^{-1} is its inverse which is also differentiable, then*

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Proof. Since f and f^{-1} are inverse to each other, we have that

$$x = f(f^{-1}(x))$$

and so taking the derivative of both sides gives

$$1 = \frac{d}{dx}[x] = \frac{d}{dx}[f\left(f^{-1}(x)\right)] = f'\left(f^{-1}(x)\right)\left(f^{-1}\right)'(x)$$

and isolating $\left(f^{-1}\right)'(x)$ gives

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

□

To show off this theorem, let's finally find the derivative of a few inverse trigonometric functions. We will denote the inverse functions as follows:

$$f(x) = \sin x \quad \Longrightarrow \quad f^{-1}(x) = \arcsin x$$

Example 3.2. *Find the derivative of the following functions.*

(1) $\arcsin x$

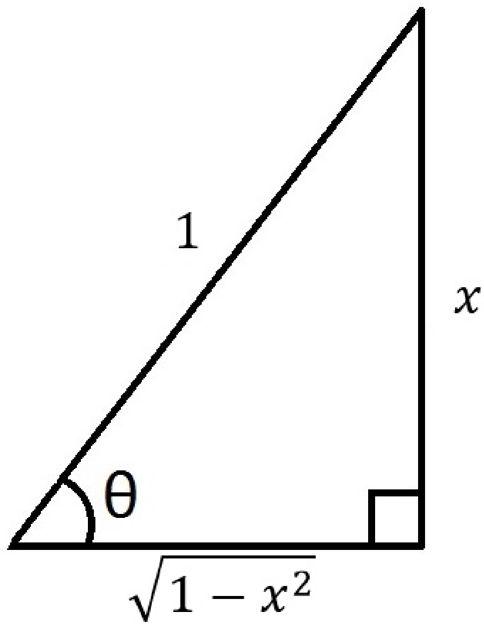
(2) $\arctan x$

Solution.

(1) Let $f(x) = \sin x$ so that $f^{-1}(x) = \arcsin x$. Then

$$\frac{d}{dx}[\arcsin x] = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\cos(\arcsin x)}$$

Using the right triangle to figure out $\cos(\arcsin x)$



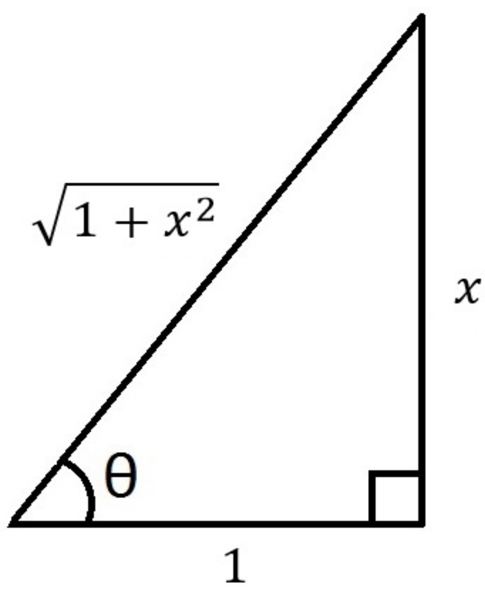
we see that $\cos(\arcsin x) = \sqrt{1-x^2}$ so that

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

(2) Let $f(x) = \tan x$ so that $f^{-1}(x) = \arctan x$. Then

$$\frac{d}{dx}[\arctan x] = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\sec^2(\arctan x)}$$

Using the right triangle to figure out $\sec^2(\arctan x)$



we see that $\sec^2(\arctan x) = \left(\sqrt{1+x^2}\right)^2 = 1 + x^2$ so that

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$$

The other derivatives of inverse trigonometric functions are

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}[\text{arccot } x] = -\frac{1}{1+x^2}$$

The domain for $y = \text{arcsec } x$ and $y = \text{arccsc } x$ is $|x| \geq 1$. The ranges for $y = \text{arcsec } x$ and $y = \text{arccsc } x$, however, are something that is not universally decided. This comes from choosing different domains for $x = \sec y$ and $x = \csc y$. Different books tend to choose different domains, for example, some texts will choose the domains to be $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ for $x = \sec y$ and $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$, while others will choose $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ for $x = \sec y$ and $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ for $x = \csc y$. We will choose the first option to make the derivative look simpler. The difference we see in the derivatives is an absolute value on the x in the denominator in the latter case.

$$\frac{d}{dx}[\text{arcsec } x] = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}[\text{arccsc } x] = -\frac{1}{x\sqrt{x^2-1}}$$

3.1.1. *Exercises.* Compute the derivatives of the following functions

(1) $\arccos x$

(2) $\text{arccot } x$

3.2. Logarithmic and Exponential Functions.

3.2.1. *Natural Logarithm.* As of yet, we have not integrated $\frac{1}{x}$. Now, we integrate it by making a definition:

Definition 3.3 (Natural Logarithm)**.** For any x in the interval $(0, \infty)$, we define the *natural logarithm* of x to be the number

$$\ln x = \int_1^x \frac{1}{t} dt.$$

This means, of course, that

$$\frac{d}{dx}[\ln x] = \frac{1}{x}.$$

We can use this to find the antiderivative of $\frac{1}{x}$ for all $x \neq 0$. Observe the following

$$\frac{d}{dx}[\ln(-x)] = \frac{1}{-x}(-1) = \frac{1}{x}$$

so that

$$\frac{d}{dx}[\ln x] = \frac{d}{dx}[\ln(-x)]$$

Hence we can define, for all $x \neq 0$

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

Remark 3.4. This definition of $\ln x$ satisfies the expected properties of logarithms: $\ln 1 = 0$, $\ln(ab) = \ln a + \ln b$, and $\ln(a^n) = n \ln a$; none of which are too difficult to check.

Now that we have the integral of $\frac{1}{x}$, we can finally integrate the rest of the trigonometric functions

Example 3.5. *Find the integral of the following functions*

(1) $\tan x$

(2) $\sec x$

Solution.

(1)

$$\begin{aligned} \int \tan x\, dx &= \int \frac{\sin x}{\cos x} dx \overset{u=\cos x}{=} \int \frac{-1}{u} du \\ &= -\ln|u| + C = -\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C \end{aligned}$$

(2)

$$\begin{aligned} \int \sec x\, dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\ &\overset{u=\sec x+\tan x}{=} \int \frac{1}{u} du = \ln|u| + C \\ &= \ln|\sec x + \tan x| + C \end{aligned}$$

The remaining integrals of trigonometric functions are

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C$$

and

$$\int \cot x\, dx = \ln|\sin x| + C$$

To integrate $\ln x$, we actually need to use integration by parts! In the integral

$$\int \ln x\, dx$$

let $u = \ln x$ and $dv = dx$, then $du = \frac{1}{x} dx$ and $v = x$ so that

$$\int \ln x\, dx = x \ln x - \int x \frac{1}{x}\, dx = x \ln x - \int dx = x \ln x - x + C$$

3.2.2. *Natural Exponential.*

Definition 3.6 (The Natural Number)**.** Let us define the *natural number*, e, to be the number such that

$$\ln e = 1$$

Definition 3.7 (Natural Exponential Function)**.** The *natural exponential function* is the function given by raising e to the power x, i.e.,

$$f(x) = e^x$$

The function e^x is the inverse function to $\ln x$ since

$$\ln(e^x) = x \ln e = x(1) = x$$

Using the fact that e^x is the inverse function to $\ln x$, we can find its derivative. Let $f(x) = \ln x$, then $f^{-1}(x) = e^x$, then applying the result for differentiating inverses gives

$$\frac{d}{dx}\left[e^x\right] = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\frac{1}{e^x}} = e^x$$

That is, e^x is its own derivative! Of course, this also implies that

$$\int e^x \, dx = e^x + C$$

since the integral is the inverse operation to the derivative.

Example 3.8. *Compute*

$$\frac{d}{dx}[e^{x \cos x}].$$

Solution.

$$\frac{d}{dx}[e^{x\cos x}] = e^{x\cos x}\frac{d}{dx}[x\cos x] = e^{x\cos x}(\cos x - x\sin x)$$

Example 3.9. *Compute*

$$\int x^2 e^{x^3} \, dx$$

Solution. This fits the pattern of a u-sub problem with $u = x^3$, so that $du = 3x^2\,dx$

$$\int x^2 e^{x^3}\,dx = \int \frac{1}{3} e^u du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$$

As a final example, here is an integration by parts example that more frequently shows up

Example 3.10. *Integrate the function* $e^x \sin x$.

Solution. In this case, following the LIATE rule, we choose $u = \sin x$ and $dv = e^x dx$. Then $du = \cos x\,dx$ and $v = e^x$, and we get

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$$

Since we still can't do that integral, we do integration by parts again, this time with $u = \cos x$ and $dv = e^x dx$. Then $du = -\sin x\,dx$ and $dv = e^x dx$ so that

$$\int e^x \sin x\,dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x\,dx\right) = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$$

Notice how we end up with the integral we started with! We can move that to the other side to get

$$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x + C$$

and then we get that the integral is

$$\int e^x \sin x\,dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + C$$

3.2.3. *Exercises.* Find the derivatives of the following functions

(1) $\ln(3x)$

(2) $\ln(2x^2+1)$

(3) $\ln(t+1)^2$

(4) $\ln\sqrt{x^2-4}$

(5) $\ln(\ln x)$

(6) $\ln\sqrt{\dfrac{x+1}{x-1}}$

(7) $\ln\left|\dfrac{\cos x}{\cos x-1}\right|$

(8) e^{2x}

(9) $e^{\sqrt{x}}$

(10) $2e^{x^2+1}$

(11) $e^x\ln x$

(12) $\ln\left(1+e^{9x}\right)$

(13) $\dfrac{e^{2x}}{e^{2x}+1}$

(14) $\displaystyle\int_{\pi}^{\ln x}\cos e^t\,dt$

Find the integrals of the following functions

(15) $\cot x$

(16) $\csc x$

(17) $\dfrac{1}{x+1}$

(18) $\dfrac{(\ln x)^2}{x}$

(19) $\dfrac{9}{5-4x}$

(20) $\dfrac{x^2}{4-5x^3}$

(21) $\dfrac{x^2+2x+3}{x^3+3x^2+9x}$

(22) $\dfrac{1}{1+\sqrt{2x}}$

(23) $e^{-x^4}(-4x^3)$

(24) e^{2x}

(25) $e^x(e^x+1)^2$

(26) $\dfrac{e^{\sqrt{x}}}{\sqrt{x}}$

(27) e^{1-x}

(28) $\dfrac{1}{e^{2x}}$